Problem Solving

by

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The main goal in Mathematics is to solve problems. The problems sometimes involve numbers as we know and sometimes they are abstract. They are all based on deductive or inductive logic. Whatever the nature the problem, there is a set of assumptions and a conclusion or a set of conclusions. Starting with the set of assumptions and a set of known facts one must logically come to a conclusion. This is called the deductive method.

The deductive method uses "if…then" logic. The student should be trained to develop a logical chain of statements from assumptions to the conclusion/s. This is called "Sufficient Condition" logic.

There is another conditional logic, "necessary condition" logic. It uses words such as "In order to …we must have…" or "…only if …" Students should also know this type of logic.

Most people get confused between the two types of logic. Basically "sufficient" means it gets the "job done." While "necessary" means it is a prerequisite but may not be enough to get the "job done." For example, a student needs 120 credits to graduate with a B.S. degree is a necessary condition, but it is not sufficient. Just any 120 credits won't be enough to graduate. The sufficient condition is that the credits are as per the requirements of the degree program.

"…Our brains aren't wired for general logic problems .." says neuroscientist David Eagleman in "Incognito – The secret lives of the brain". That is the reason we have a difficult time at solving puzzles and word problems. The only way to get adept at solving word problems is to solve as many of them as possible. Then the process becomes automatic. "Burn really good programs all the way to the DNA," says David Eagleman.

Mathematical Problem Solving

A mathematics problem is usually well-defined and involves necessary and sufficient conditions. The only thing we need to do is to establish logical links.

> As an illustration we will take up a typical problem in Algebra and follow it through various steps.
>
> Sara invested $10,000 in two funds, one fund that earned 9% and one that earned 7%. If she earned $880 in interest from the two funds, how much did Sara invest in each fund?

Here the goal is to find out how much Sara invested in each fund; the <u>necessary conditions</u> are how much she invested in the two funds at what rates, and how much the total interest is. The <u>sufficient conditions</u> are the information provided in first and second sentence of the problem. Thus the problem is complete with necessary and sufficient conditions. If we omitted one of the pieces of these lines, the information would not be sufficient.

This problem is solved in an elaborated manner later, but here we will use it to explain various steps of problem solving.

The steps to solve a word problem are as follows and depend on sufficiency logic.

The steps are based on assumptions or given information.

1. Read the problem, verbalize it in your words to comprehend what is given and what is asked, labeling the unknowns in letters (variables) if they exist.

   For the example given above, given: investment $10,000 in two funds at rates 9% and 7%. The total interest is $880.00. One unknown investment is x at 9%.

2. Translate the given information into a mathematical equation or inequality using the given.

   If the investment at 9% is x, the investment at 7% is 10000 – x.

   The total interest in terms of variables is .09 x + .07(10000 – x) while the numerical value is 880. The main point is to set up an equation, we have .09x + .07(10000 – x) = 880.

3. Check the tools you have learned in the section or the book, and use them to solve the equation.

   Simplify the terms on the left side of the equation and solve.

   .09x + 700 - .07x = 880 (Remove parentheses)

   .02x = 180 (Isolate the variable one side and numbers on the other.)

   2x = 18000 (We multiplied by 100.)

   x = 9000 (We divided by 2)

4. Once you solve the equation, translate back the values of variables in terms of the values of the quantities.

   The investment at 9% is $9,000 and at 7% it is $1,000.

Thus, a problem is solved going one step at a time from "what is known" (given) to "what is unknown," using "If…then…." All arithmetic problems and most algebra problems can be solved using this "if…then…" connections or sufficient condition logic. First find out the unknown and call it x or some variable. From the first line start connecting the sentences until the last "if…then…" The solution is found by isolating the equation formed in the process.

In all arithmetic word problems and most simple algebra problems, 'If …then…" logic works. But in some challenging algebra problems, another approach called "Ambitious Target Tree" needs to be used. Building "an ambitious target tree" involves starting from labeling the unknown and going back listing various obstacles. Once we list the obstacles, we list the Intermediate Objectives (IO's). For each intermediate objective, some action is necessary. While doing this, we find two equal quantities which give rise to an equation which we must solve. In the following all problems from the monograph, "Singapore Model Method" are solved using TOC Thinking Process tools. Only the arrows for if … then are not drawn.

After studying these problems, the student will be ready to solve any word problem in Mathematics, or start proving theorems, which are abstract word problems.

There is also another method using a "transition tree."

There are three examples in the last part of this handout using the three methods: sufficient condition, Ambitious Target Tree and transition tree. These three problems are not in the Singapore Model Method monograph. The example of how to solve a problem involving a system of equations that uses a transition tree is also essential in solving an abstract word problem (proving a theorem.)

Singapore Model Method - Word Problems

Arithmetic Problems: Read each sentence, translate into symbols (analyze), and connect using if ... then ... logic.

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Example 1: (Primary 4B) 48 children went to the zoo. 3/8 of them were girls. How many boys were there?

Solution: If (48 children went to the zoo **and** 3/8 of them were girls), then 3/8 of 48 were girls.

3/8*48 = 18
18 out of 48 were girls. 48 - 18 = 30 were boys.

Example 2: (Primary 5A) Peter collected a total of 1170 stamps. He collected 4 times as many Singapore stamps as foreign stamps. How many Singapore stamps did he collect?

Solution: Singapore stamps and foreign stamps total 1170 stamps. Singapore stamps = 4 times foreign stamps. If Singapore stamps and foreign stamps total 1170 stamps and Singapore stamps are four times foreign stamps), then the number of stamps is 5 times the foreign stamps, and Peter has 1170/5 = 234 foreign stamps. Then Peter has 4*234 = 936 Singapore stamps.

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Example 3: (Primary 5A) Mrs. Lin made 300 tarts. She sold 3/4 of them and gave 1/3 of the remainder to her neighbor. How many tarts had she left?

Solution: If Mrs. Lin made 300 tarts and she sold ¾ of them, then she sold 3/4 *300 = 225. She was left with 75 tarts. She gave 1/3 of the remainder (75) to her neighbor. i.e. she gave 1/3*75 = 25 to her neighbor. She had 75 -

25 = 50 tarts left.

Example 4: (primary 6B) Meiling spent an equal amount of money each day. After 4 days, she had 4/5 of her money left. After another ten days, she had $30 left. How much money did she have at first?

Solution: Start from the Goal (last line) and go backwards.

Obstacles	IO's	Action
Don't know the total on the first day	Assume some variable amount on the first day.	T dollars
Don't know how much money she had spent after 14 days.	Find the amount spent.	T -30 dollars.
Don't know the amount she had left after 4 days	Find the amount left after 4 days.	4/5 T dollars
Don't know how much she spent in four days.	Find it.	1 1T - 4/5 T = 1/5 T 2 4*money daily spent
Don't know how much she spent every day.	Find the amount.	After 14 days she spent T - 30 dollars. She spent equal amount each day. Therefore, she spent 1/14(T - 30) every day.
Don't know how to set up an equation.	Find two equal quantities for the money spent in 4 days.	1/5 T = 4* 1/14 (T - 30)
Don't know how to solve the equation.	Isolate T.	Multiply each side by lcd = 70 and simplify. 14 T = 20(T - 30) 14T = 20T - 600 -6T = - 600 T= 100

In this problem one has to be careful about the money left and money spent. Here we read the first column as follows: In order to find the total on the first day, I must find how much she spent after 14 days… etc. Notice we are going backwards in the problem.

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Example 5: (Primary 6B) Raju had 3 times as much money as Gopal. After Raju had spent $60 and Gopal had spent $10, they each had an equal amount of money left. How much money did Raju have at first?

Solution: Read each line and analyze.

Raju had 3 times as much money as Gopal.

If the amount with Gopal is x, then the amount with Raju = 3x.

Second line. Amount with Raju - 60 = Amount with Gopal - 10

Translating in symbols, 3x - 60 = x - 10

Solve for x: $3x - x = 60 - 10$
$2x = 50$

Raju had 75 dollars at first.

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Example 6 (Primary 6B): 3/5 of the beads in a box are yellow beads. The rest are red and blue beads. There are twice as many yellow beads as red beads. There are 30 more red beads than blue beads. Find the total number of yellow and red beads.

Solution: Read and analyze, translate each line into symbols:

First line: If the number of total beads is x, then the yellow beads are 3/5 x.

Second line: Then the number of red and blue beads = the rest = x – 3/5x = 2/5x

Third line: Then number of red beads = ½*yellow beads = 1/2 * 3/5 *x = 3/10 x

Fourth line: Then the number of blue beads = the number of red beads - 30 = 3/10 x - 30

The total number of beads = x = yellow beads + red beads + blue beads

$$x = 3/5\ x + 3/10\ x + (3/10x - 30)$$

Multiply by lcd 10 and Isolate x: $10\ x = 6x + 3x + 3x - 300$
$-2x = -300$
$x = 150$

Total number of beads = 150.

The number of yellow and red beads = 3/5 x + 3/10 x = 3/5*150 + 3/10*150 = 90 + 45 = 135.

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Example 7 (Primary 6B): The ratio of Sumin's money to Meili's money was 4:1. After Sumin had spent $26. Sumin had $2 less than Meili. How much money did Sumin have at first?

(Information: If a ratio is given the actual fraction is obtained by multiplying a variable x to the numerator and denominator of the ratio.)

Solution: Ratio 4:1 means actual amounts are 4x and 1x = x. Sumin had 4x dollars and Meili had x dollars

Second line: After Sumin spent $26 he had 4x - 26 dollars.

Third line: If Sumin had $2 less than Meili, then Sumin had x - 2 dollars.

If the second line and third line is true, then we have an equation: $4x - 26 = x - 2$

Simplifying and solving for x: $3x = 24$. $x = 8$. Meili had \$8 and Sumin had 4*8 dollars = \$32.

Example 8 (Primary 6 B): The ratio of Peter's money to John's money was 3:5 at first. After Peter's money was increased by \$250 and John's money was decreased by by \$350, they had an equal amount of money. How much money did Peter have at first?

Solution: Ratio 3:5 means actual amounts are 3x and 5x. Peter had 3x dollars and John had 5x dollars.
Line 2: After Peter's money was increased by \$250, Peter had 3x + 250 dollars. After John's money was decreased by \$350, John had 5x - 350.

Line 3: They had equal amounts of money: $3x + 250 = 5x - 350$
Solving for x, we get $2x = 600$. $x = 300$. Peter had 3*300 = \$900 at first.

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Example 9 (Primary 6B): Jenny and Marvin have 836 stamps altogether. Jenny has 20% more stamps than Marvin. How many more stamps does Jenny have than Marvin?

Solution: Read and analyze line by line.

Line 2: If Jenny has 20% more stamps than Marvin, and Marvin has x stamps, then Jenny has x + 20%x = x + .2x stamps.

i.e. 2.2x = 836 Solving, x = 836/2.2 = 380 Marvin had 380 stamps. Jenny has .2x = .2*380 = 76 stamps more than Marvin.

Integration of the Model Method and Algebra

Page 53

Example 1 There are 50 children in a dance group. If there are 10 more boys than girls, how many girls are there?

Solution: Line 2 first part: If there are 10 more boys than girls, and there are x girls then there are x + 10 boys.
Line 1: If the total is 50, then x + 10 + x = 40. Solving for x: 2x = 40, x = 20 girls.

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Example 2 A has 3 times as much money as B. B has \$200 less than C. C has \$50 more than A. Find the total amount of money that A, B and C have.

Solution: Line 2: If B has \$200 less than C, then B's money = *C's* money - 200.

Line 3: If C has \$50 more than A, then *C's* money = A's money + 50.

If Line 2 and line 3 is true, then B's money = (A's money + 50) - 200.

Line 1: If A has 3 times as much money as B, then A's money = 3*B's money = 3*((A's money + 50)- 200) = 3*(A's money -150)

i.e. A's money = 3*A's money - 450

Solving for A's money: 2*A's money = 450. A has $225. From line 3, C has $275. From line 2, B has $75.

Total amount is 225 + 275 + 75 = $575

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Example 3 Mary had spent 3/5 of her money on a dictionary. She spent another 1/3 of her money on a pen. She spent $84 altogether. How much money did she have at first?

Solution: Assume Mary had x dollars at first.

Line 1: She spent 3/5 of x on a dictionary. i.e. 3/5 x on a dictionary.

Line 2: She spent 1/3 x on a pen. Line 3: She spent a total of 1/3 x + 3/5 x = 84 dollars.

Simplifying: lcd = 15. Multiplying by 15, 5x + 9x = 1260

i.e. 14x = 1260

i.e. x = 1260/14 = 90

Mary had $90 at first.

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Example 4 It is given that the number of quarters and dimes are in the ratio of 2:3. If 4 of the quarters are exchanged for dimes, the ratio will become 2:7. What is the total value ($) of the set of coins?

Solution: Information: Ratio 2:3 means actual amounts are 2x and 3x. There are 2x quarters and 3x dimes.

Line 2: 4 quarters are exchanged for dimes. This means you make a change of 4 quarters in dimes. You get 10 dimes.

There are 4 less quarters and 10 more dimes. i.e. 2x - 4 quarters and 3x + 10 dimes. They are in the ratio 2:7.

i.e. (2x - 4)/(3x + 10) = 2/7

i.e. cross-multiplying: 7(2x - 4) = 2(3x + 10)

Simplifying, 14x - 28 = 6x + 20

8x = 48

i.e. x = 6

There are 12 quarters and 18 dimes. The total amount in dollars is (600+360).01 = $9.60

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Example 5 Jenny and Marvin have 836 stamps altogether. Jenny has 20% more stamps than Marvin. How many more stamps does Jenny have than Marvin? (Same as Example 9, Page 50)

Solution: It is necessary to understand that Marvin's stamps should be taken as the basis. Marvin has x stamps. If Jenny has 20% more stamps than Marvin's, then Jenny has .2 times x + x stamps = $.2x + x$ = 1.2x stamps.

Algebra Problems

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Example 1: There are 50 children in a dance group. If there are 10 more boys than girls, how girls are there?

Solution:
Line 1: Boys + girls = 50 i.e. x + 10 + x = 50 or

2x + 10 = 50

2x = 40 or x = 20 girls.

Example 2: $120 is shared among 3 persons A, B and C. If A receives $20 less than B, and B receives 3 times as much money as C, how much money does C receive?

Solution: Line 2 (first part): A receives $20 less than B. A's money = B's money - 20.

Line 2 (second part): B receives 3 times as much money as C. B's money = 3* C's money.

Line 2 and Line 3 imply A's money = 3*C's money - 20

Line 1: Total = A's money + B's money + C's money = 3*C's money - 20 + 3*C's money + C's money = 120

i.e. 7*C's money - 20 = 120

i.e. 7*C's money = 140

or C's money = $20

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Example 16: A sum of money is shared among three persons A, B and C so that A receives 10% more than B, and B receives 10% more than C. If A receives $525 more than C, find the sum of money.

Solution: Don't know how much C receives: Let x be the amount C receives.

B receives 10% more than C: B gets x + .1x = 1.1 x

A receives 10% more than B: A gets 1.1x + .1(1.1x) = x(1.1)(1 +.1) = 1.1^2x = 1.21 x

A receives 525 more than C: 1.21 x = 525 + x

Solve the equation: .21 x = 525

x = 525/.21 = 52500/21 = 2500

The total is (1 + 1.1+ 1.21)x = 3.31(2500) = $ 8275

Example 17: There are twice as many boys as girls in a choir. If the number of boys is decreased by 30%, by what percentage must the number of girls be changed so that there will be an equal number of boys and girls in the choir?

Solution:

Line 1: The ratio of boys to girls is 2:1. This means the there are 2x boys and x girls in the choir.

Line 2: If the number of boys is decreased by 30%, then we have 2x - .3(2x) boys in the choir. The number of girls in the choir is 2x (1 -.3) = 2x *.7 = 1.4x.

This is more than x the original number of girls. The increase is 1.4x - x = .4x or fraction is .4x/x = .4 or 40%.

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Example 18 The ratio of the number of boys to the number of girls is 5:6. It is given that 30% of the boys wear glasses, and there is an equal number of boys and girls who wear glasses. What percentage of the girls wear glasses?

Solution:

If the ratio is 5:6, then the total number of boys is 5x and total number of girls is 6x. 30% of boys wear glasses means .3(5x)= 1.5x boys wear glasses.

Equal number of boys and girls wear glasses: the number of girls who wear glasses is 1.5 x. The fraction of girls wearing glasses is 1.5x/6x = 1.5/6 = 1/4

i.e. 25%

Solving Challenging Algebra Word Problems

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Example 1: A certain amount of water is poured from a jug into an empty mug so that the amount of water in the mug is 1/6 the amount of water left in the jug. If 50 ml of water is further poured from the jug to the mug, the amount of water in the mug will be 1/5 of that left in the jug. Find the original amount of water in the jug.

Solution: Here one has to be careful about the jug and the mug.

Obstacles	IO's	Action
Don't know the amount in the mug.	Assume the amount in the mug.	y ml is the amount in the mug.
Don't know the amount of water left in the jug.	Find the amount in the jug	6y ml
Don't know the amounts after the second transfer.	Find the amount in the jug and in the mug.	6y - 50 ml in the jug y + 50 in the mug
Don't know the relation between the amount of water in the mug and jug.	Find it.	y + 50 = 1/5{6y - 50)
Don't know how to solve the equation.	Find and multiply the common denominator	Multiply by 5. 5y + 250 = 6y - 50 300=y
Don't know the amount of water in the jug.	Find the amount.	After the first transfer, left over is 6y, poured amount is y. 6y + y is the total amount. 2,100 ml

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Example 2: At a musical concert, class A tickets were sold at $4 each, class B tickets at $2 each and souvenir programs at $1 each. 3/4 of the audience who bought class A tickets and 2/3 of the audience who bought class B tickets also bought the programs. The total amount of money collected from both types of tickets was $1400 and the amount of money collected from the programs was $350. Find the total number of people who attended the concert.

Solution:

Obstacles	IO's	Action
Don't know Class A tickets, Class B tickets sold	Assume the numbers	Assume Class A: x, Class B: y tickets sold.
Don't know the amount for tickets of the two classes.	Find the amounts.	Amount for tickets of Class A: 4x, Amount for tickets of Class B: 2y
Don't know the programs for each type.	Find the amounts.	3/4 x , 2/3y
Don't know the total from programs.	Find the amounts.	At $1 ea., 3/4 x + 2/3 y
Don't know the total amount of money collected from tickets.	Find the amounts and set up equation.	4x + 2y = 1400
Don't know the total amount from programs.	Find the equation.	3/4x + 2/3 y = 350
Don't know the values of x and y.	Solve the system.	4x + 2y = 1400, 9x + 8y = 4200 Multiply by 4 to get rid of y: 16x + 8y = 5600, 9x + 8y = 4200 7x = 1400, x = 200, Class A tickets y = 300, Class B tickets
Don't know the total.	Add the separate amounts.	500 people.

Three Methods of Solving Mathematical Word Problems:

Sufficient condition, Ambitious Target Tree, Transition Tree

Sufficient Condition Method:

Example 1: (From the Singapore Model Method Monograph, problem for Primary 6B – names changed) Karen had 3 times as much money as Jill. After Karen had spent \$60 and Jill had spent \$10, they each had an equal amount of money left. How much money did Karen have at first?

Solution: Read each line and analyze.

Karen had 3 times as much money as Jill.

If the amount with Jill is x, then the amount with Karen = 3x.

Second line. Amount with Karen - 60 = Amount with Jill - 10

Translating in symbols, 3x - 60 = x - 10

Solve for x: 3x - x = 60 - 10
2x = 50

Karen had 75 dollars at first.

Ambitious Target Tree Method

Example 2: (From the Singapore Model Method Monograph, problem for Primary 6B) Meiling spent an equal amount of money each day. After 4 days, she had 4/5 of her money left. After another ten days, she had \$30 left. How much money did she have at first?

Solution: Start from the Goal (last line) and go backwards.

Obstacles	IO's	Action
Don't know the total on the first day	Assume some variable amount on the first day.	T dollars
Don't know how much money she had spent after 14 days.	Find the amount spent.	T -30 dollars.
Don't know the amount she had left after 4 days	Find the amount left after 4 days.	4/5 dollars
Don't know how much she spent in four days.	Find it.	1. 1T - 4/5 T = 1/5 T 2. 4*money daily spent
Don't know how much she spent every day.	Find the amount.	After 14 days she spent T - 30 dollars. She spent equal each day. Therefore, she 1/14(T - 30) every day.

Don't know how to set up an equation.	Find two equal quantities for the money spent in 4 days.	1/5 T = 4* 1/14 (T - 30)
Don't know how to solve the equation.	Isolate T.	Multiply each side by lcd = 70 and simplify. 14 T = 20(T - 30) 14T = 20T - 600 -6T = - 600 T= 100

In this problem one has to be careful about the money left and money spent. Here we read the first column as follows: "In order to find the total on the first day, I must find how much she spent after 14 days. In order to find how much money she had spent after four days, we need to find how much she had left after 4 days.... etc. "
Notice we are going backwards in the problem.

Solving a word problem using a transition tree (TrT).

The following example is a typical word problem in an algebra text.

Example 3: A woman had $10,000 that she invested in two funds, one fund that earned 9% and one that earned 7%. If she earned $880 in interest from the two funds, how much was invested in each fund?

Solution: Usually a figure or a chart helps to analyze the problem. We usually do not draw entities and arrows, but in this example we will. First we will solve the problem without using entities and then using a TrT.

Given (Assumptions): The woman invested $10,000, some of which is at 9% and the rest at 7%. If x is the amount invested at 9%, then the rest of the amount is 10000 – x at 7%, because, both investments must add up to 10000. (x + (10000 – x) = 10000)

The interest of each investment is given by: .09 x and .07(10000 –x). The total interest in algebraic terms is .09x + .07(10000 – x), and the total given interest is $880. We equate them.

$$.09x + .07(10000 - x) = 880$$

To solve this equation we get rid of decimals by multiplying every term by 100 to get

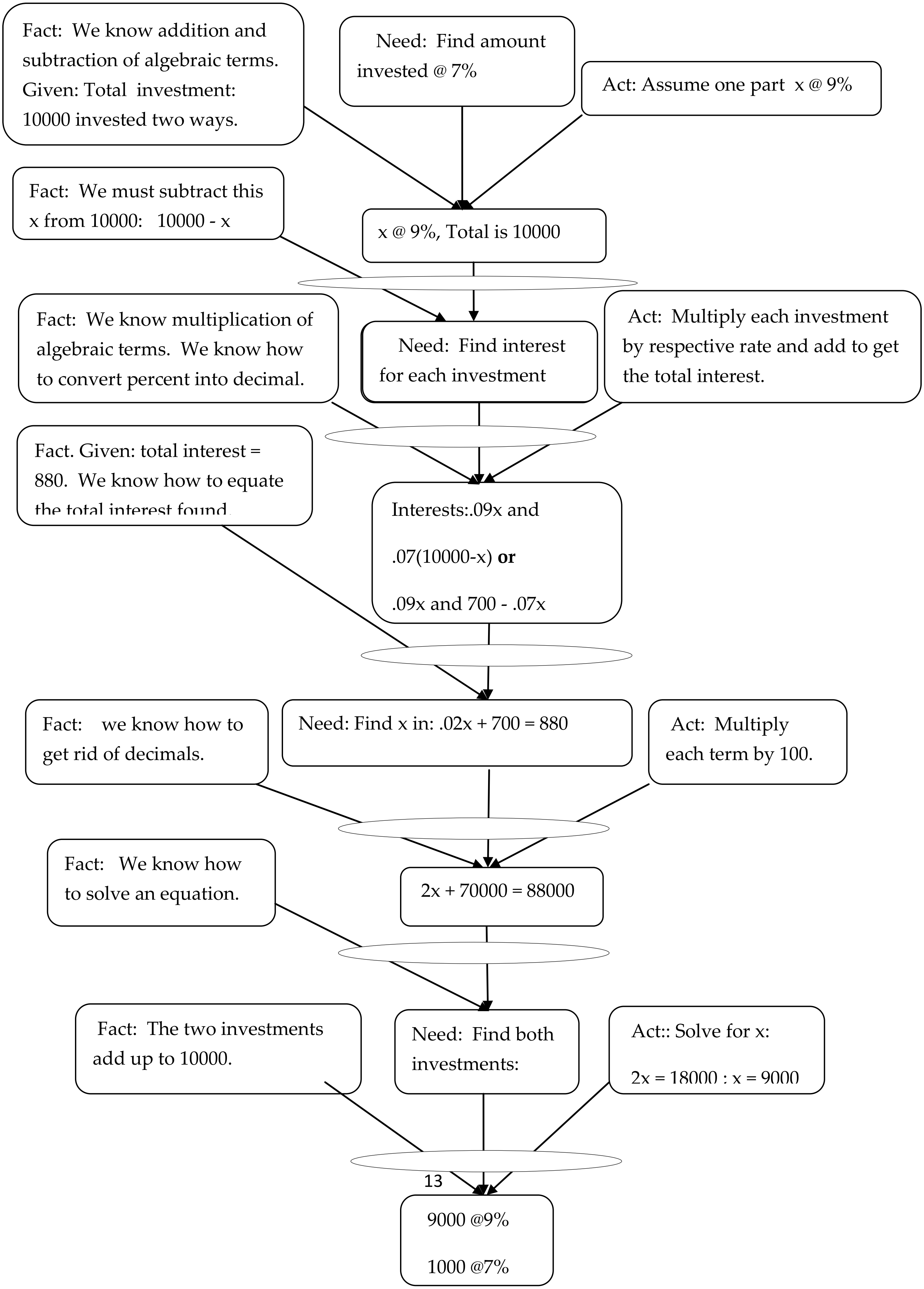
$9x + 7(10000 - x) = 88000$

$9x + 70000 - 7x = 88000$ removing parentheses

$2x = 18000$ simplifying

$x = 9000$

The woman invested $9000@9% and 1000@7%


Fact: We know addition and subtraction of algebraic terms. Given: Total investment: 10000 invested two ways.
Need: Find amount invested @ 7%
Act: Assume one part x @ 9%
Fact: We must subtract this x from 10000: 10000 - x
x @ 9%, Total is 10000
Fact: We know multiplication of algebraic terms. We know how to convert percent into decimal.
Need: Find interest for each investment
Act: Multiply each investment by respective rate and add to get the total interest.
Fact. Given: total interest = 880. We know how to equate the total interest found.
Interests:.09x and
.07(10000-x) **or**
.09x and 700 - .07x
Fact: we know how to get rid of decimals.
Need: Find x in: .02x + 700 = 880
Act: Multiply each term by 100.
Fact: We know how to solve an equation.
2x + 70000 = 88000
Fact: The two investments add up to 10000.
Need: Find both investments:
Act:: Solve for x:
2x = 18000 : x = 9000
9000 @9%
1000 @7%

Example of another TrT to solve a systems of three equations in three unknowns

Suppose the student wants to solve word problems using systems of three linear or non-linear equations in three unknowns.

Here is a set of actions obtained by constructing a "Transition Tree" See figures of steps on next pages for complete justification.

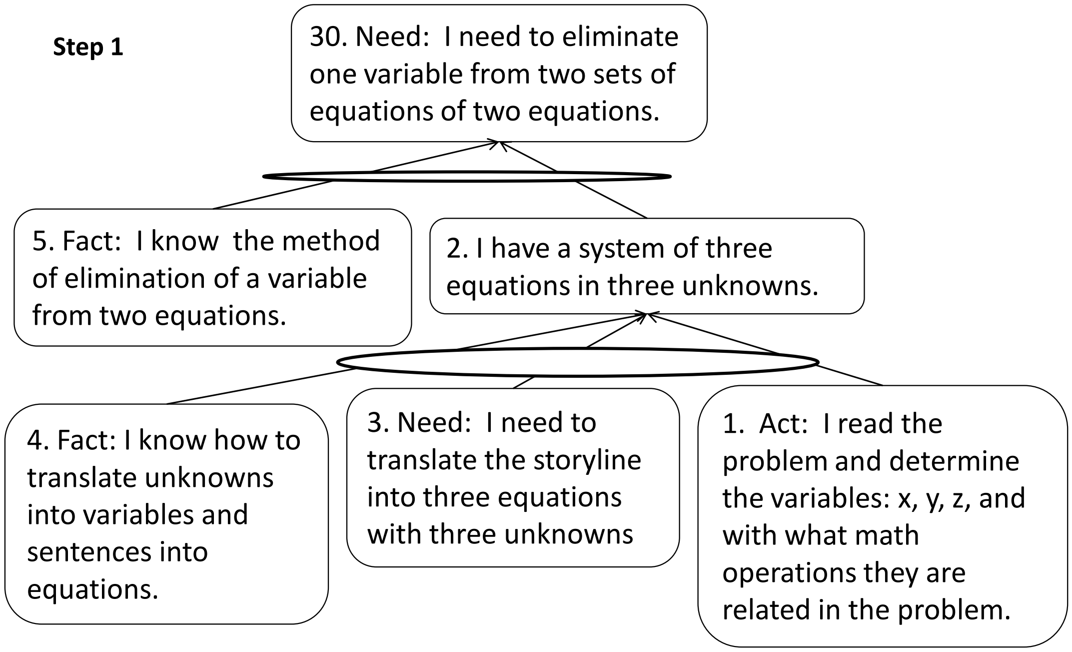
1. I read the word problem identifying the unknown quantities as: x, y, z.
2. I translate sentences into equations. I must have three equations.
3. Rearranging if necessary, I use the first equation and one of the methods – substitution or addition/subtraction - to eliminate the first variable (x, y or z) from the remaining equations. I get two equations in two variables. This is the second step. I now proceed to eliminate the second variable from the two equations at the second step repeating the procedure and I reach the last step which reduces to
    1) an equation of one variable, or
    2) has zeroes on both sides or
    3) has a zero on the left side and a non-zero number on the right side.
4. If I have 1) an equation of one variable, I solve it and use the value of that variable to find the value of the second variable from one of the equations at the second step. I then use the two values of the two variables to find the value of the third variable. I have a unique solution(s) for each variable. Once I find the values of all variables, I interpret the meaning in the given context.

   If in the third step, if I have 2) the last equation has all zeroes, then the system has "infinitely many solutions."

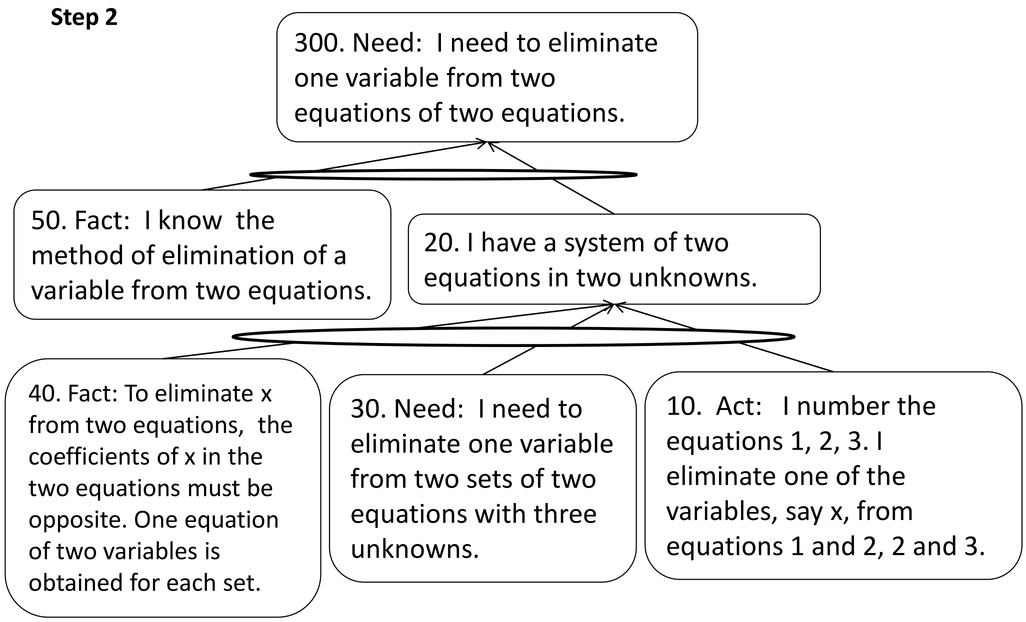
   In the third step an equation has zero on the left side and a nonzero number on the right side. In this case, there is no solution and the system is called "inconsistent."

Transition Tree – To solve a word problem with three equations in three unknowns. The problem is done in three steps.

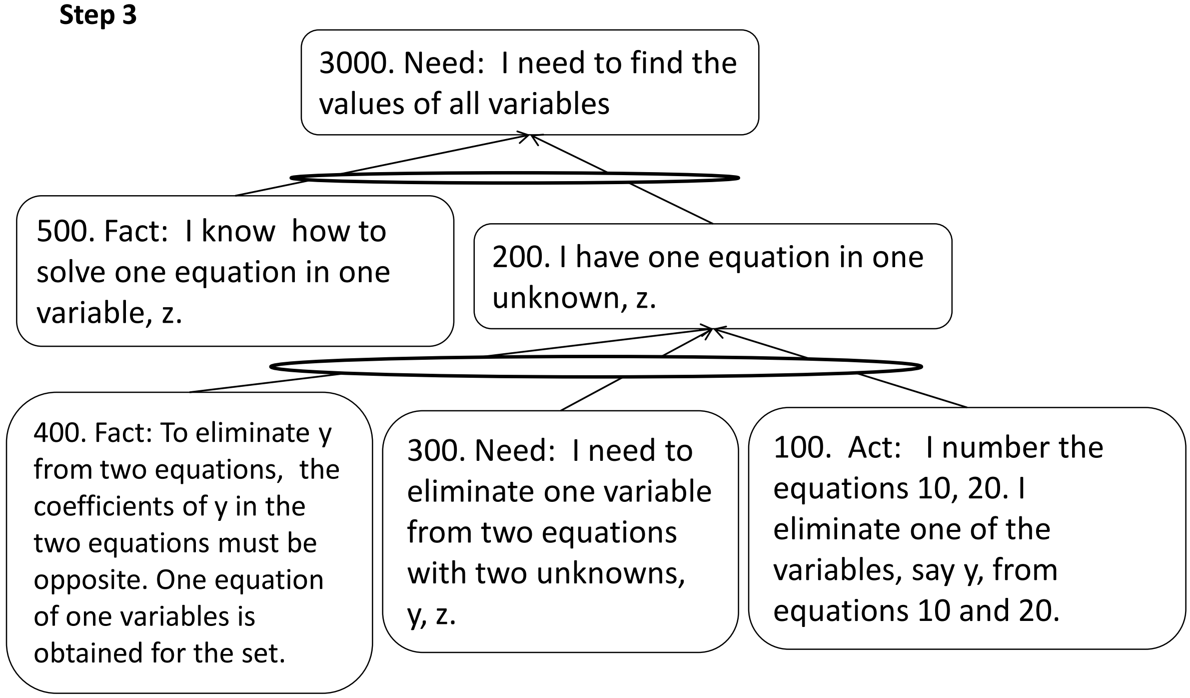
Step 1



Step 2



Step 3



Examining the step 200 may result in three scenarios discussed earlier. Only one scenario of getting a genuine equation is considered in Step 3.